

# Analytical Solutions for Open Channel Temperature Response to Unsteady Thermal Discharge and Boundary Heating

H. S. Tang<sup>1</sup> and T. R. Keen<sup>2</sup>

**Abstract:** Analytical solutions are derived for a one-dimensional model of the bulk temperature response of open channel flow with unsteady and nonuniform heating at an upstream boundary, the water surface, and the riverbed. The model describes the temperature variation as kinematic waves, and the solutions are explicit formulas that are comprised of transient terms, which play dominant roles at the upstream end, and equilibrium terms, which determine the temperature far downstream. It is shown that the time-dependence of the solutions includes solution envelopes that can be both complex as well as interesting because of the interplay between the upstream and lateral boundary conditions. The applicability of the solutions to practical problems is demonstrated for two cases: (1) a stream bounded at its upstream end by a dam and with a mid-reach inflow that is also subject to diurnal heating; and (2) Boulder Creek, Colorado, which is impacted by effluent released from a wastewater treatment plant. The model prediction is in reasonable agreement with gauged data.

**CE Database subject headings:** Analytical techniques; Water temperature; Open channel flow; Wave propagation.

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Report Documentation Page				Form Approved OMB No. 0704-0188	
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>2009</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2009 to 00-00-2009</b>	
4. TITLE AND SUBTITLE <b>Analytical Solutions for Open Channel Temperature Response to Unsteady Thermal Discharge and Boundary Heating</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Naval Research Laboratory, Oceanographic Division, Code 7320, Stennis Space Center, MS, 39529-5004</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES <b>Journal of Hydraulic Engineering vol 135 327-332</b>					
14. ABSTRACT <b>see report</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>39</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

## Introduction

Although accurate and computationally affordable numerical simulations are becoming the dominant approach for predicting mass and heat transfer in various flows, analytical approaches continue to be used because of their irreplaceable roles in many studies (e.g., Taylor 1921, Fischer *et al.* 1979, Tang and Sotiropoulos 1999, van Dongeren and Svendsen 2000, Chan *et al.* 2006). This paper presents a model for the bulk temperature of water bodies such as rivers, lakes, and reservoirs that have time-dependent thermal forcing at their upstream ends and water surface, side bank, and bed (hereinafter referred to as boundaries ). The model is based on an earlier model described by Edinger *et al.* (1974) and Jobson and Schoellhammer (1987) that, in addition to both variants and related techniques, has been applied in natural streams (e.g., Kim and Chapra 1997; Boyd and Kasper 2003; Westhoff *et al.* 2007), variable reservoir releases (e.g., Carron and Rajaram 2001), and river temperature control (e.g., Gu *et al.* 1999). A comprehensive description, including supporting parameters for phenomena such as air-water heat exchange, can be found in Edinger *et al.* (1974), Jobson and Schoellhammer (1987), and Martin and McCutcheon (1999). It should be noted, however, that the governing equations of the model are typically solved numerically.

This study considers that the flow is fully mixed over its cross-section and that diffusion is negligible. The temperature response is physically simulated as a series of one-dimensional kinematic waves subject to unsteady thermal boundary conditions. The boundary conditions at the upstream and lateral boundaries are either periodic or arbitrary functions of time and space. The resulting governing equation is a first-order linear

differential equation that can be analytically solved. It consists of an unsteady term, a convective term, and a source term. The solutions of the equation are direct extensions of the equilibrium solution of Edinger *et al.* (1974), which is valid far from an upstream boundary. It should be noted that analytical solutions for similar equations have been derived in other studies (e.g., Chatwin 1973; van Genuchten and Alves 1982; Shukla 2002; Weigand 2004), which consider longitudinal diffusion and dispersion of a flow with an initial condition and unsteady upstream boundary condition. However, these studies did not consider time-dependent heating and cooling along the channel. Despite the simplified governing equations, the solutions presented in this paper can provide estimates for practical environmental problems such as the construction of a dam.

## Conceptual Model and Governing Equations

The model is depicted in Fig. 1, which illustrates a one-dimensional flow in a channel with variable-temperature water entering its upstream end and heat input along its length. Assuming that the flow is well mixed over its cross-section and ignoring diffusion, the governing equation for the bulk temperature can be represented using the principles of thermal balance as follows (e.g., Edinger *et al.* 1974; Jobson and Schoellhammer 1987)

$$\frac{\partial T(t, s)}{\partial t} + u(s) \frac{\partial T(t, s)}{\partial s} = K(T_{bd}(t, s) - T(t, s)) \quad 0 < s < \infty \quad (1)$$

$$T(t, s) = T_{up}(t) \quad s = 0 \quad (2)$$

where:  $t$  is the time;  $s$  is the distance downstream;  $T(t, s)$  is the temperature,  $T_{up}(t)$  is the temperature at the upstream boundary,  $T_{bd}(t, s)$  is the lateral boundary temperature, and  $u(s)$  is the flow velocity; both temperature and velocity are cross-section averages. The bulk coefficient of heat transfer,  $K$ (defined later), is dependent on wind speed, flow depth, and heat capacity of the water body (Edinger *et al.* 1974; Jobson and Schoellhammer 1987). In this investigation,  $K$  is a constant. The time-dependent temperature at the upstream end is given by Eq. (2), and the unsteady thermal input at lateral boundaries is given by the right hand side (RHS) of Eq. (1).

Eq. (1) is a linear partial differential equation that describes the propagation of temperature waves. In order to complete the general problem stated by Eqs. (1) and (2), an initial condition for temperature is needed. This study considers only equilibrium solutions, which are independent of the initial temperature condition, , and thus no initial condition is used. We will present three solutions of Eqs. (1) and (2), each corresponding to a different set of thermal boundary conditions and flow velocity.

### ***Case 1: Constant velocity flow with sinusoidal time-dependent thermal discharge and boundary heating***

Case 1 has the following flow and temperature conditions

$$u(s) = U \quad (3)$$

$$T_{up}(t) = \bar{T}_0 + T_0 \sin(\omega_0 t + \alpha) \quad (4)$$

$$T_{bd}(t, s) = \bar{T}_1 + T_1 \sin \omega_1 t \quad (5)$$

where:  $U$  is a constant velocity;  $\bar{T}_0$ ,  $T_0$ ,  $\omega_0$ , and  $\alpha$  are the average, amplitude, frequency, and phase of the upstream temperature fluctuation, respectively; and  $\bar{T}_1$ ,  $T_1$ , and  $\omega_l$  are the average, amplitude, and frequency of temperature at the lateral boundaries, respectively.  $\bar{T}_0$ ,  $T_0$ ,  $\omega_0$ ,  $\alpha$ ,  $\bar{T}_1$ ,  $T_1$ , and  $\omega_l$  are constants. Conditions (4) and (5) are frequently used as approximations of measurements (e.g., Velz 1970; Edinger *et al.* 1974). For example, Eq. (4) corresponds to the temperature modulation of water released from detention ponds and reservoirs, whereas Eq. (5) represents diurnal heating at the water surface. Case 1 is a direct extension of the problem discussed by Edinger *et al.* (1974), who used Eqs. (1), (3), and (5) to describe the thermal response of natural water bodies.

### ***Case 2: Flow with spatially and time-dependent boundary heating***

Case 2 is an extension of Case 1 but with a spatially variable lateral thermal boundary condition. The flow velocity and upstream temperature are represented by Eqs. (3) and (4), but a spatial and time-dependent temperature is used at lateral boundaries

$$T_{bd}(t, s) = \bar{T}_1 + \bar{T}_1' s + (T_1 + T_1' s) \sin \omega_l t \quad (6)$$

where:  $\bar{T}_1'$  and  $T_1'$  are the downstream gradients of average temperature and amplitude, respectively. Condition (6) can represent temporal and spatial changes of atmospheric temperature, riverbed temperature, groundwater discharge, and snowmelt at higher elevations (e.g., Hanrahan 2007, Westhoff 2007).

### ***Case 3: Flow with spatially variable velocity and time-dependent upstream temperature and boundary heating***

As another extension of Case 1, Case 3 incorporates a spatially variable flow velocity and time-dependent temperature at upstream and lateral boundaries

$$u(s) = V(s) \quad (7)$$

$$T_{up}(t) = H(t) \quad (8)$$

$$T_{bd}(t, s) = \bar{T}_1 + \sum_{i=1} T_i \sin(\omega_i t + \beta_i) \quad (9)$$

The right hand side of Eq. (9) defines an arbitrary function of time if the Fourier trigonometric series converges. Therefore, the flow velocity, inflow temperature at an upstream boundary, and lateral boundary temperature are arbitrary functions of distance or time.

## **Solutions and Discussion**

### **Case 1**

In view of Lagrangian coordinates,  $T(t, s)$  may be considered as the temperature of a control cross-section by letting  $s = S(t)$ ,  $S(t)$  being the traveling distance of the cross-section. The problem represented by Eqs. (1) - (5) can then be transformed to

$$\frac{dT(t, S(t))}{dt} + KT(t, S(t)) = KT_{bd}(t, S(t)) \quad 0 < t < \infty \quad (10)$$

$$T(t, S(t)) = T_{up}(t) \quad t = t_0 \quad (11)$$

where:  $t_0$  is a reference time, and  $dS(t)/dt = U$ , or,  $t_0 = t - S(t)/U$ . Eqs. (10) and (11) comprise an initial value problem for a linear, first-order, non-homogeneous ordinary differential equation. The solution can be obtained as follows (see Polyanin and Zaitsev 1995)

$$T(t, s) = \bar{T}_1 + \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin(\omega_1 t - \theta_1) + \left[ \bar{T}_0 - \bar{T}_1 + T_0 \sin\left\{ \omega_0 \left( t - \frac{s}{U} \right) + \alpha \right\} - \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin\left\{ \omega_1 \left( t - \frac{s}{U} \right) - \theta_1 \right\} \right] \cdot \exp\left\{ -\frac{Ks}{U} \right\} \quad (12)$$

where  $\theta_1 = \tan^{-1}(\omega_1 / K)$ . Condition (4) has been used to eliminate  $t_0$  in obtaining Eq. (12). Eq. (12) is the solution for Case 1 and will be referred to as the base solution hereinafter.

The base solution consists of two parts: (1) the first two terms on the RHS of Eq. (12), which reflect the influence of the lateral boundary condition, are the equilibrium solution far downstream as discussed by Edinger *et al.* (1974); and (2) the transient solution represented by the exponential term on the RHS, which results from both the upstream and lateral boundary conditions. The two parts and their interplay determines the behavior of the solution. Only the first part dominates the solution far downstream because the second part is damped.



The Case 1 temperature solution given by Eq. (12) (Fig. 2) oscillates with decreasing amplitude until the equilibrium region is reached far downstream. The solution exhibits a transient stage with a spatial scale that is about the same whether the upstream temperature is higher or lower than the equilibrium . Consequently, a characteristic transient length  $L_T$  can be defined

$$L_T = \frac{U}{K} \quad (13)$$

$L_T$  describes the effects of the upstream temperature; the transient component will reduce to 5% ( $e^{-3}$ ) in strength at a distance downstream of  $3L_T$ , which is located at  $sK/U=3$  in Fig. 2.

The solution envelope for Case 1 (see Fig. 2) is also damped downstream, but the damping is not monotonic. The cause of this behavior can be explained as follows. If the frequency of the upstream temperature oscillation is the same as that of the lateral boundary temperature ( $\omega_0 = \omega_1$ ) the base solution, Eq. (12), becomes

$$T(t, s) = \bar{T}_1 + (\bar{T}_0 - \bar{T}_1) \cdot \exp\left\{-\frac{Ks}{U}\right\} + A \sin(\omega_1 t - \theta_1 + \kappa) \quad (14)$$

where

$$A = \sqrt{\left( B \exp\left\{-\frac{Ks}{U}\right\} - \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \right)^2 + \frac{2BKT_1}{\sqrt{K^2 + \omega_1^2}} \left( 1 + \cos\left(-\frac{\omega_1 s}{U} + \theta_1 + \gamma\right) \right) \cdot \exp\left\{-\frac{Ks}{U}\right\}}, \quad (15)$$

$$B = \sqrt{T_0^2 + \frac{K^2 T_1^2}{K^2 + \omega_1^2} - \frac{2KT_0 T_1}{\sqrt{K^2 + \omega_1^2}} \cos(\alpha - \theta_1)}, \quad (16)$$

$$\gamma = \tan^{-1} \frac{T_0 \sin \alpha - (KT_1 / \sqrt{K^2 + \omega_1^2}) \sin \theta_1}{T_0 \cos \alpha - (KT_1 / \sqrt{K^2 + \omega_1^2}) \cos \theta_1}, \quad (17)$$

$$\kappa = \tan^{-1} \frac{B \exp\{-Ks/U\} \sin(-\omega_1 s/U + \theta_1 + \gamma)}{KT_1 / \sqrt{K^2 + \omega_1^2} + B \exp\{-Ks/U\} \cos(-\omega_1 s/U + \theta_1 + \gamma)}. \quad (18)$$

where  $A$  is the amplitude of the solution envelope. This term oscillates downstream because of the cosine term in Eq. (15), but it is slowly damped because of the exponential term. If we define a characteristic envelope length as

$$L_E = \frac{U}{\omega_1} \quad (19)$$

it can be seen from Eq. (15) that the envelope wavelength is  $2\pi L_E$ , which is determined by the velocity and temperature frequency. The envelope in Fig. 2 has a spatial wavelength of  $2\pi L_E K / U = 0.84$ .

An interesting situation occurs in applications such as river temperature control when the fluctuating component of the upstream temperature, as given in Eq. (4), is equal to the equilibrium temperature far downstream;  $T_0 = KT_1 / \sqrt{K^2 + \omega_1^2}$ ,  $\omega_0 = \omega_1$ , and  $\alpha = -\theta_1$ . For example, using the parameters of Fig. 2a and setting the upstream temperature fluctuation in this way, the resulting solution reveals that the envelope appears as parallel rather than wavy lines (Fig. 3). In fact, the base solution, Eq. (12), now becomes

$$T(t, s) = \bar{T}_1 + \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin(\omega_1 t - \theta_1) + [\bar{T}_0 - \bar{T}_1] \cdot \exp\left\{-\frac{Ks}{U}\right\} \quad (20)$$

The amplitude of the temporal term (second term on the RHS) is a constant and thus the extremes of the solution will not oscillate, and the temperature envelope is defined by parallel lines.

Upstream and downstream time histories (Fig. 4) of the Case 1 solution in Fig. 2a demonstrate the superimposed effects of the upstream and lateral boundary conditions. The resulting upstream temperature oscillation consists of multiple frequencies whereas the downstream solution is dominated by the periodic heating and has a single frequency.

## Case 2

After a tedious calculation similar to that for Eq. (12), the solution of Case 2 becomes

$$\begin{aligned}
 T(t, s) = & \bar{T}_1 + \bar{T}_1' s - \frac{U\bar{T}_1'}{K} + \frac{K(T_1 + T_1' s)}{\sqrt{K^2 + \omega_1^2}} \sin(\omega_1 t - \theta_1) - \frac{KUT_1'}{K^2 + \omega_1^2} \sin(\omega_1 t - \theta_1') \\
 & + \left[ \bar{T}_0 - \bar{T}_1 + \frac{U\bar{T}_1'}{K} + T_0 \sin\left\{ \omega_0 \left( t - \frac{s}{U} \right) + \alpha \right\} - \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin\left\{ \omega_1 \left( t - \frac{s}{U} \right) - \theta_1 \right\} \right. \\
 & \left. + \frac{KUT_1'}{K^2 + \omega_1^2} \sin\left\{ \omega_1 \left( t - \frac{s}{U} \right) - \theta_1' \right\} \right] \cdot \exp\left\{ -\frac{Ks}{U} \right\}, \quad (21)
 \end{aligned}$$

where  $\theta_1' = \tan^{-1}(2\omega_1 K / (K^2 - \omega_1^2))$ . This solution will reduce to the base solution of Eq. (12) under conditions (3), (4), and (5). The solution of Eq. (21) with the flow velocity and upstream boundary condition for temperature given in Fig. 2a, but with temporally and spatially variable lateral boundary heating, is plotted in Fig. 5. Physically, this boundary condition produces an increased downstream temperature compared to Fig. 2a.

## Case 3

In this situation,  $dS(t)/dt = V(S(t))$ , or,  $t_0 = t - \int_0^{S(t)} (1/V(r))dr$ . By a straightforward

derivation similar to that for the base solution, the solution is obtained as

$$\begin{aligned}
T(t, s) = & \bar{T}_1 + \sum_{i=1} \frac{KT_i}{\sqrt{K^2 + \omega_i^2}} \sin(\omega_i t + \beta_i - \theta_i) \\
& + \left[ H\left(t - \int_0^s \frac{dr}{V(r)}\right) - \bar{T}_1 - \sum_{i=1} \frac{KT_i}{\sqrt{K^2 + \omega_i^2}} \sin\left\{\omega_i\left(t - \int_0^s \frac{dr}{V(r)}\right) + \beta_i - \theta_i\right\} \right] \\
& \cdot \exp\left\{-K \int_0^s \frac{dr}{V(r)}\right\},
\end{aligned} \tag{22}$$

where  $\theta_i = \tan^{-1}(\omega_i / K)$ . The behavior of Eq. (22) is illustrated with three examples that use the parameters given in Fig. 2a, but with different flow velocity, upstream temperature, and lateral boundary condition. The first example (Fig. 6a) demonstrates the effect of a spatially variable flow velocity. The second example (Fig. 6b) illustrates a different upstream temperature oscillation and Fig. 6c employs a more complex lateral boundary condition.

## Applications

### ***Open channel flow with time-dependent discharge and surface heating***

The solution of Eq. (21) is now applied to a stream or river section that is bounded at its upstream end by a dam with continuous discharge but variable water temperature. The stream also has diurnal heating. The river's mean cross-sectional temperature can be approximated by a lateral boundary condition that is not only a sinusoidal function of time, but also a spatial variable because of a change in air temperature along its length. The temperature at the dam is specified as having the same frequency as that of the

surface heating but with a phase delay. The frequency is  $1/24 \text{ hr}^{-1} = 7.27 \times 10^{-5} \text{ s}^{-1}$ . The river just downstream from the dam has a mean flow velocity of  $1 \text{ m s}^{-1}$ , a water depth of 5 m, a mean temperature of  $15^\circ\text{C}$ , and a daily fluctuation of  $3^\circ\text{C}$ . The air has an average temperature of  $20^\circ\text{C}$  at the dam and  $25^\circ\text{C}$  at 500km downstream with a linear increase, an amplitude of  $15^\circ\text{C}$  and a phase lag of  $-1.5 \text{ rad}$ .

This problem requires calculation of the heat transfer coefficient,  $K$ , which Edinger *et al.* (1974) derived as

$$K = \frac{\bar{K}}{\rho C_p h} \quad (23)$$

where:  $\rho$  is water density ( $1000 \text{ kg m}^{-3}$ );  $\bar{K}$  is the air-water exchange coefficient related to factors including wind speed ( $100 \text{ W m}^{-2} ^\circ\text{C}^{-1}$ );  $C_p$  is the heat capacity of water ( $4186 \text{ J kg}^{-1} ^\circ\text{C}^{-1}$ ); and  $h$  is the water depth (5 m). Based on these parameters,  $K$  equals  $4.8 \times 10^{-6} \text{ s}^{-1}$ . The analytical solution is plotted in Fig. 7, with the upstream and lateral boundary conditions given in the figure caption. As estimated by Eq. (19), the envelope wavelength  $2\pi L_E$  is 86 km. The transient length  $3L_T$  is obtained from Eq. (13) as 625 km, suggesting that a river with these characteristics would not approximate the equilibrium within the distance shown in the figure.

Consider a multiple branch flow; suppose the channel has a mid-reach inflow at  $s = 200$  km that raises the temperature by  $5^\circ\text{C}$  and reduces the average flow speed to  $0.5 \text{ m s}^{-1}$ . The solution for the section starting from the location of the mid-reach discharge to its downstream terminus becomes ( $s \geq 200$ )

$$T(t, s) = \bar{T}_1 + \bar{T}_1' (s - 200) - \frac{U\bar{T}_1'}{K} + \frac{T_1}{\sqrt{K^2 + \omega_1^2}} \sin(\omega_1 t - \theta_1) \\ \left[ H(t) + 5 - \bar{T}_1 + \frac{U\bar{T}_1'}{K} - \frac{KT_1}{\sqrt{K^2 + \omega_1^2}} \sin\left\{ \omega_1 \left( t - \frac{s - 200}{U} \right) - \theta_1 \right\} \right] \cdot \exp\left\{ -\frac{K(s - 200)}{U} \right\}, \quad (24)$$

where  $H(t)$  equals  $T(t, 200)$ , which is given by Eq. (21). Eq. (24) is obtained from Eq. (21) by replacing its upstream temperature terms with  $[H(t)+5]$ . The solution of Eq. (24) is also shown in Fig. 7, from which it is seen that the solution is significantly changed downstream from the mid-reach discharge because of the additional inflow.

### ***Creek flow with effluent from a wastewater treatment plant***

The analytical solution of Eq. (22) is applied to Boulder Creek, Colorado, which is impacted by effluent discharge from a wastewater treatment plant. The flow is shallow, highly transient, and well mixed. A complete description of the flow and a comprehensive 24-hour survey can be found in Windell *et al.* (1988). The creek is 13.7 km long and 0.2 to 0.6 m deep, the velocity ranges from 0.12 to 0.4 m s<sup>-1</sup>, and the creek temperature varies with time. The air temperature, wind speed, solar radiation, and other conditions change diurnally. The measurements of creek temperature were made at four stations located at 0.6 km, 5.0 km, 9.0 km, and 13.7 km downstream from the plant.

Using the measured flow rates and creek cross-sectional areas (Windell *et al.*, 1988), it is obtained that  $u = 0.27, 3.10, 0.13$ , and  $0.13$  m s<sup>-1</sup> at station 0.6, 5.0, 9.0, and 13.7 km, respectively. From the survey, it is also determined that  $T_{bd} = 15 + 10 \sin(7.27 \times 10^{-5} t)$  °C

( $t$  is in seconds),  $\bar{K} = 35 \text{ W m}^{-2} \text{ }^{\circ}\text{C}^{-1}$ , and  $h = 0.4 \text{ m}$ . Here  $T_{bd}$  is obtained by approximating the effects of air temperature and solar radiation. Linear distributions of velocity are assumed between adjacent stations and Eq. (23) is used for the heat transfer coefficient. Letting station 0.6 km to be the upstream end and approximating the observed temperature with  $T_{up} = 17 + 3 \sin(7.27 \times 10^{-5} t - 1.5) \text{ }^{\circ}\text{C}$  (Fig. 8a), the solution of Eq. (22) for the mean temperature at the other stations is given in Figs. 8b, 8c, and 8d. It is seen that the analytical solution is a reasonable estimate of the temperature fluctuations measured at the rest of stations; the analytical solution predicts ranges and phases for the creek temperature similar to the observations. The largest difference between the prediction and the measurement occurs at furthest downstream station (Fig. 8d). Although the temperature predicted by the analytical solution is not as accurate as that from a numerical model by Kim and Chapra (1997), the errors of the two approaches are of the same magnitude in comparison with the measurement. The results are promising given that the flow and temperature are affected by many factors that the analytical solution excludes, such as the material comprising the creek bed, the thermal interaction between water and sediment, and the three-dimensional character of the flow.

## Concluding Remarks

This paper presents a one-dimensional temperature response model and its analytical solutions for flows with a time-dependent upstream boundary condition for temperature, and temporally and spatially variable heating along lateral boundaries. The solutions are



expected to be useful in providing insight into the physics of the flows, conducting parameter studies, and setting up benchmark solutions for testing simulation software. As illustrated in the previous section, the solutions produce reasonable estimates of the equilibrium temperature distribution for real streams.

The analytical solutions are valid for single branches of rivers. However, as shown in the first example of the previous section, they can be applied to multiple branch rivers. In order to account for flow variability and boundary conditions, a channel can be divided into several sub-reaches with representative values for the temperature variation and heating; the solutions can then be applied to the individual sub-reaches. The analytical solutions could also be useful in estimating contaminant transport problems such as that in a stream receiving detention water (e.g., Wang *et al.* 2004). In addition to open channel flows, it is anticipated that the solutions are applicable to heat and mass transfer in other areas (e.g., Morro 1977; Maiani 2003), as long as they are one-dimensional kinematic wave phenomena. The analytical solutions are explicit formulas consisting of simple functions with a number of controlling parameters, but the calculations of the solutions can be tedious. We propose as future work to make an Internet accessible solution tool, such as a small computer code or Excel spread sheet, which allows users to easily set up inputs and obtain solutions.

It should be noted that the analytical solutions are designed for equilibrium when a balance is achieved between the time-dependent temperature of inflow at upstream ends and temporally and spatially variable heating along lateral boundaries. The solutions will

not be applicable when the effects of the initial temperature distributions are important. Moreover, they exclude diffusion effects, which are often important in practice. In this regard, it is expected that the solutions' prediction could be problematic. For instance, they will tend to predict larger temperature extremes as diffusion increases in importance. Also, the solution accuracy is limited because factors such as evaporation and the three-dimensionality of the problems are neglected. To include these factors, a numerical approach must be adopted (e.g., Jobson and Schoellhammer 1987; Kim and Chapra 1997; Tang *et al.* 2008).

## **Acknowledgement**

The first author was partially supported by Pacific Northwest National Laboratory and National Research Council Research Associateship Award. The second author was supported by the Office of Naval Research through program element 0601153N. Discussions with Drs. J. E. Edinger, P. Roberts, M. Malik, and Z. Yang are acknowledged. The authors are grateful to the editor and the anonymous reviewers for their valuable suggestions.

## Notation

The following symbols are used in this paper:

$A$	= amplitude in Eq. (14);
$B$	= parameter in Eq. (15);
$C_p$	= heat capacity;
$h$	= water depth;
$H(t)$	= time-dependent upstream boundary condition for temperature;
$L_E$	= characteristic envelope wavelength;
$L_T$	= characteristic transient length;
$i$	= integers, $\geq 1$ ;
$K$	= coefficient of heat transfer;
$\bar{K}$	= air/water exchange coefficient;
$s$	= distance downstream;
$S(t)$	= traveling distance of cross-section;
$t$	= time;
$t_0$	= reference time;
$T_0$	= upstream temperature oscillation amplitude;
$\bar{T}_0$	= average upstream inflow temperature;
$T_1$	= lateral boundary oscillation amplitude for temperature;
$T_1'$	= downstream gradient of the amplitude of the lateral boundary condition temperature;

$T_i$	= amplitude of lateral boundary condition temperature due to component $i$ ;
$\bar{T}_1$	= mean lateral boundary condition for temperature;
$\bar{T}_1'$	= downstream gradient of mean lateral boundary temperature;
$T(t,s)$	= mean cross-sectional temperature;
$T_{bd}(t,s)$	= lateral boundary condition for temperature;
$T_{up}(t)$	= inflow temperature at an upstream boundary;
$u(s)$	= cross-sectional mean flow velocity;
$U$	= a constant flow velocity;
$V(s)$	= a variable flow velocity;
$\alpha$	= phase of temperature oscillation at an upstream boundary;
$\beta_i$	= phase of lateral boundary condition oscillation for temperature due to component $i$ ;
$\omega_0$	= frequency of temperature oscillation at an upstream boundary;
$\omega_1$	= frequency of temperature oscillation at a lateral boundary;
$\omega_i$	= frequency of lateral boundary temperature oscillation due to component $i$ ;
$\theta_1$	= phase in Eq. (12);
$\theta_i$	= phases in Eq. (22);
$\theta_1'$	= phase in Eq. (21);
$\gamma, \kappa$	= phases in Eqs. (14) and (15);
$\rho$	= water density.

## References

- Boyd, M., and Kasper, B. (2003). "Analytical methods for dynamic open channel heat and mass transfer: methodology for Heat Source Model Version 7.0." Watershed Sciences Inc., Portland, Oregon. <http://www.deq.state.or.us/wq/TMDLs/tools.htm>
- Carron, C. J., and Rajaram, H. (2001). "Impact of variable reservoir releases on management of downstream water temperature." *Water Resources Research*, 37, 1733-1743.
- Chan, A. T., Lai, N. T. N, and Siddheshwar, P. G. (2006). "Analytical study of turbulence pollutant dispersion near a low hill." *J. Mech. Eng.*, 132, 99-106.
- Chatwin, P. C. (1973). "On the longitudinal dispersion of dye whose concentration varies harmonically with time." *J. Fluid Mech.*, 58, 657—667.
- Edinger, J. E., Brady, D. K., and Geyer, J. C. (1974). "Heat exchange and transport in the environment." Johns Hopkins Univ.
- Fischer, H. B., List, E. J., Koh, R. C. Y., Imberger, J., and Brooks, N. A. (1979). *Mixing in inland and coastal waters*, Academic Press, Inc., New York.
- Gu, R.C., McCutcheon, S., and Chen, C. J. (1999). "Development of weather-dependent flow requirements for river temperature control." *Environmental Management*, 24, 529-540.
- Hanrahan, T. P. (2007). "Large-scale spatial variability of riverbed temperature gradients in Snake River fall Chinook salmon spawning areas." *River Res. Applic.*, 23, 323-341.

- Jobson, H. E., and Schoellhammer, D. H. (1987). "Users manual a branched Lagrangian transport model: US Geological Survey Water-Resources Investigations." 87-4163.
- Kim, K.S., and Chapra, S.C. (1997). "Temperature model for highly transient shallow streams." *J. Hydr. Eng.*, 123, 30-40.
- Martin, J.L., and McCutcheon, S.C. (1999). *Hydrodynamics and Transport for Water Quality Modeling*, CRC Press, Boca Raton, FL, USA.
- Morro, A. (1977). "Temperature waves in rigid materials with memory." *Meccanica*, 12, 73-77.
- Maiani, A., de Kruijf, W. J. M., and Ambrosini, W. (2003). "An analytical model for the determination of stabilities boundaries in a natural circulation single-phase thermosyphoon loop." *Heat and Flow*, 24, 853-863.
- Polyanin, A.D., and Zaitsev, V. F. (1995). *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press, Inc., Boca Raton.
- Shukla, V. P. (2002). "Analytical solution for unsteady transport dispersion of nonconservative pollutant with time-dependent periodic waste discharge concentration." *J. Hydr. Eng.*, 128, 866-869.
- Tang, H.S., Paik, J., Sotiropoulos, F., and Khangaonkar, T. "Three-dimensional numerical modeling of dynamic initial mixing of multi-port thermal discharges." *J. Hydr. Eng.*, in press.
- Tang, H. S. and Sotiropoulos, F. (1999) "A second-order Godunov method for wave problems in coupled solid-water-gas systems." *J. Comput. Phys.*, 151, 790-815
- Taylor, G. I. (1921). "Diffusion by continuous movements." *Proceedings of the London Mathematical Society*, 20, 196-211.

- van Dongeren, A. R., and Svendsen, I. A. (2000). "Nonlinear and 3D effects in leaky infragravity waves." *Coastal Engineering*, 41, 467-496.
- van Genuchten, M. T., and Alves, W. J. (1982). "Analytical solutions of one-dimensional convective-dispersive solute transport equation." *Technical Bulletin 1661*, U.S. Dept. of Agriculture, Agriculture Research Service.
- Velz, C. (1970). *Applied Stream Sanitation*, Wiley-Interscience, New York.
- Wang, G.-T., Chen, S., Barber, M.E., and Yonge, D.R. (2004). "Modeling of flow and pollutant removal of wet detention pond treating stormwater runoff." *J. of Environ. Eng.*, 130, 1315-1321.
- Weigand, B. (2004). *Analytical Methods for Heat Transfer and Fluid Flow Problems*, Springer, Berlin, New York.
- Westhoff, M. C., Savenije, H. H. G., Luxemburg, W. M. J., Stelling, G. S., van de Giesen, N. C., Selker, J. S., Pfister, L., and Uhlenbrook, S. (2007). "A distributed stream temperature model using high resolution temperature observations." *Hydrol. Earth Syst. Sci.*, 11, 1469-1480.
- Windell, J. T., Rink, L. P., and Knud-Hansen, C. F. (1988). "A 24-hour synoptic water quality study of Boulder Creek between 75<sup>th</sup> Street wastewater treatment plant and Coal Creek." Aquatic and Wetlands Consultants Inc., Boulder, CO.

## Figure Captions

Fig. 1. Schematic representation of the flow.

Fig. 2. Base solution of Eq. (12).  $U=0.1 \text{ m s}^{-1}$ ,  $\bar{T}_1=50 \text{ }^\circ\text{C}$ , and  $K=0.04 \text{ s}^{-1}$ .

$$T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t) . \text{ (a) } T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5) . \text{ (b)}$$

$$T_{up}/\bar{T}_1 = 1.6 + 0.2 \sin(2t + 1.5)$$

Fig. 3. Base solution of Eq. (12) with the upstream boundary condition determined by the equilibrium temperature far downstream.  $U=0.1 \text{ m s}^{-1}$ ,  $\bar{T}_1=50 \text{ }^\circ\text{C}$ , and  $K=0.04 \text{ s}^{-1}$ .

$$T_{up}/\bar{T}_1 = 0.4 + 0.0326 \sin(0.3t - 1.438) , \quad T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t)$$

Fig. 4. Time histories of the base solution of Eq. (12) at upstream and downstream locations.  $U=0.1 \text{ m s}^{-1}$ ,  $\bar{T}_1=50 \text{ }^\circ\text{C}$ , and  $K=0.04 \text{ s}^{-1}$ .  $T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5)$ ,

$$T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t)$$

Fig. 5. Solution of Eq. (21).  $U=0.1 \text{ m s}^{-1}$ ,  $\bar{T}_1=50 \text{ }^\circ\text{C}$ , and  $K=0.04 \text{ s}^{-1}$ .  $T_{bd}/\bar{T}_1 =$

$$1 + 0.06s + (0.4 + 0.06s) \sin(0.3t) , \text{ and } T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5)$$



Fig. 6. Solution of Eq. (22).  $U=0.1$  m/s,  $\bar{T}_1=50$  °C, and  $K = 0.04$  s<sup>-1</sup>. (a)

$$u(s)/U = 1 + 0.3 \sin(3\pi s/5), T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5), \text{ and } T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t).$$

(b)  $u(s)/U = 1$ ,  $T_{up} = 0.4 + 0.04(1 + \sin(2t))^2$ , and  $T_{bd} = 1 + 0.4 \sin(0.3t)$ . (c)  $u(s)/U = 1$ ,

$$T_{up}/\bar{T}_1 = 0.4 + 0.2 \sin(2t + 1.5), \text{ and } T_{bd}/\bar{T}_1 = 1 + 0.4 \sin(0.3t) + 0.2 \sin(0.1t - 1)$$

Fig. 7. Temperature response of open-channel flow with a mid-reach inflow.  $U = 1$  m s<sup>-1</sup>,

$$T_{up} = 15 + 3 \sin(7.27 \times 10^{-5} t - 1.5) \text{ °C}, T_{bd} = 20 + 10^{-5} s + 15 \sin(7.27 \times 10^{-5} t) \text{ °C},$$

$\bar{T}_1' = 10^{-5}$  °C m<sup>-1</sup> and  $K = 4.8 \times 10^{-6}$  s<sup>-1</sup>.  $t$  and  $s$  are in seconds and meters, respectively. An

inflow is located at  $s=200$  km, where it raises temperature of the flow by 5 °C and

reduces the velocity to  $U = 0.5$  m s<sup>-1</sup> ( $s \geq 200$  km)

Fig. 8. Temperature response of a creek flow with effluent from a wastewater treatment

plant. Define Error =  $\left[ (1/n) \sum_{i=1}^n [(T^i - T_m^i)/T_m^i]^2 \right]^{1/2}$ , where  $n$  is total number of

measurements during the 24 hrs, and  $T_m^i$  and  $T^i$  are respectively the  $i$ th measurement

data and the corresponding predicted temperature. (a) Station located at 0.6 km,

Error=0.047. This location is set up as the upstream end, and the solid line in the figure is

actually the upstream condition of the analytical model. (b) Station located at 5.0 km,

Error=0.098. (c) Station located at 9.0 km, Error=0.056. (d) Station located at 13 km,

Error=0.13



Figure  
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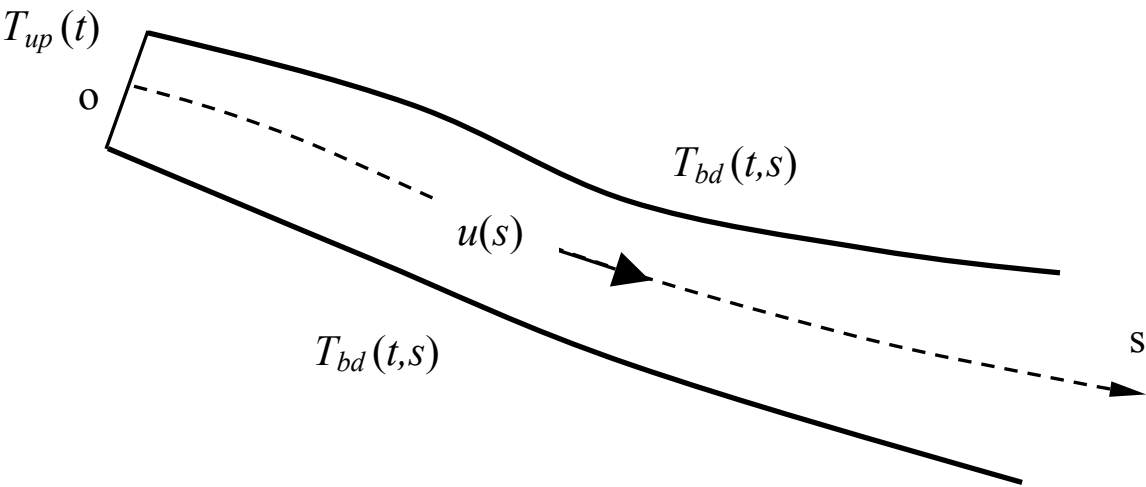


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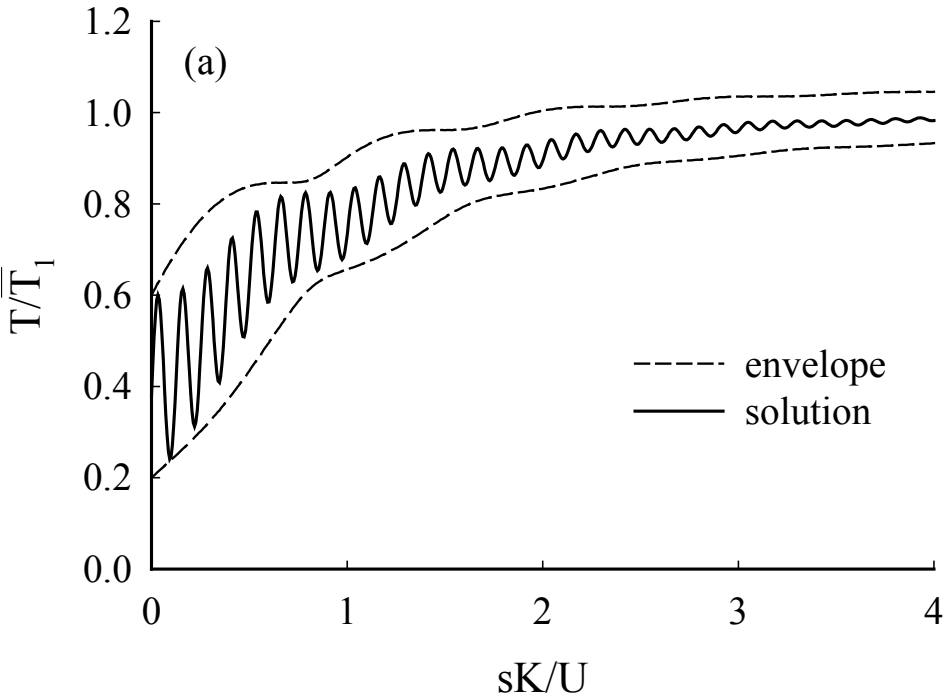


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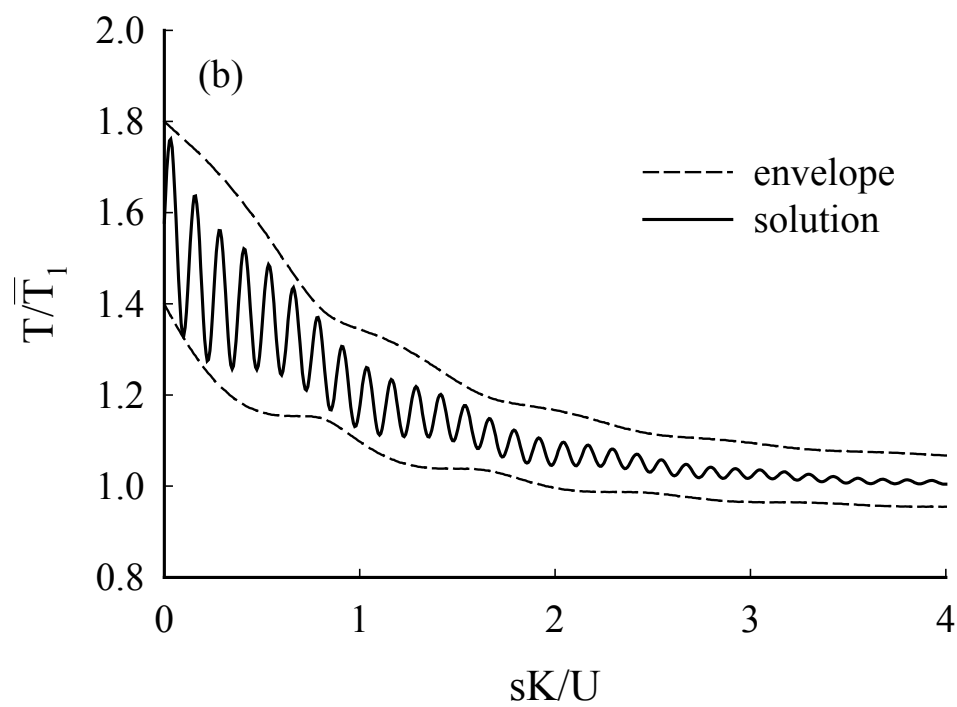


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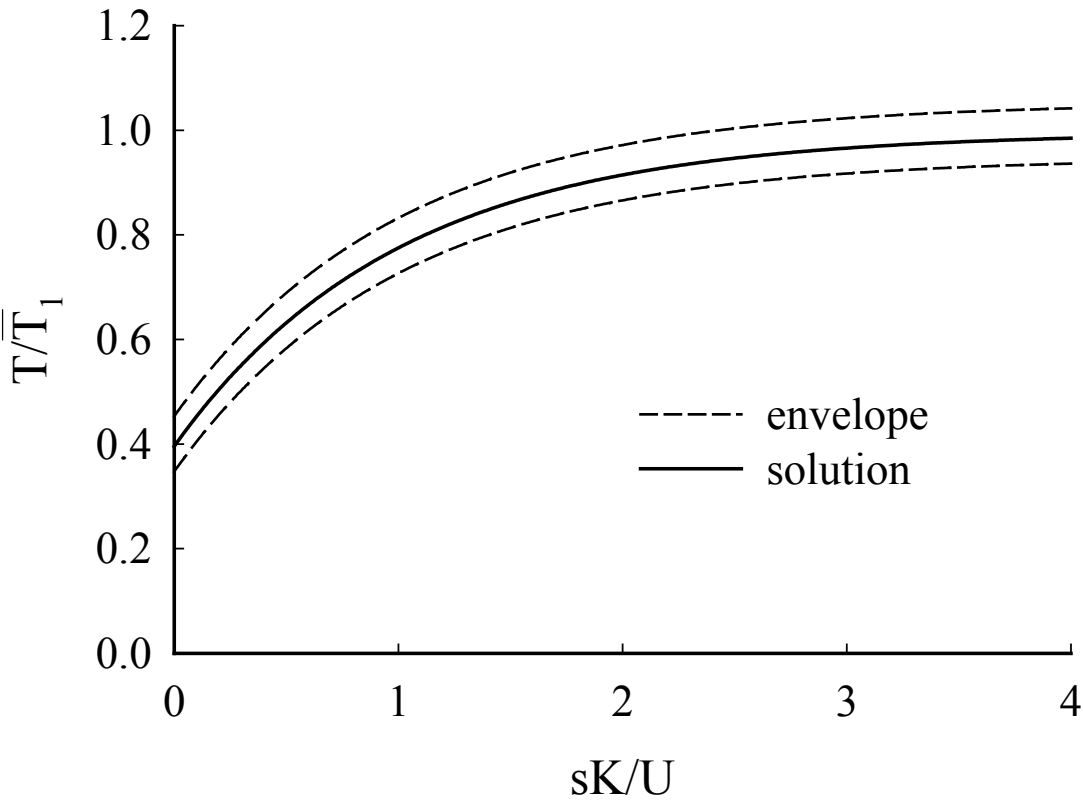


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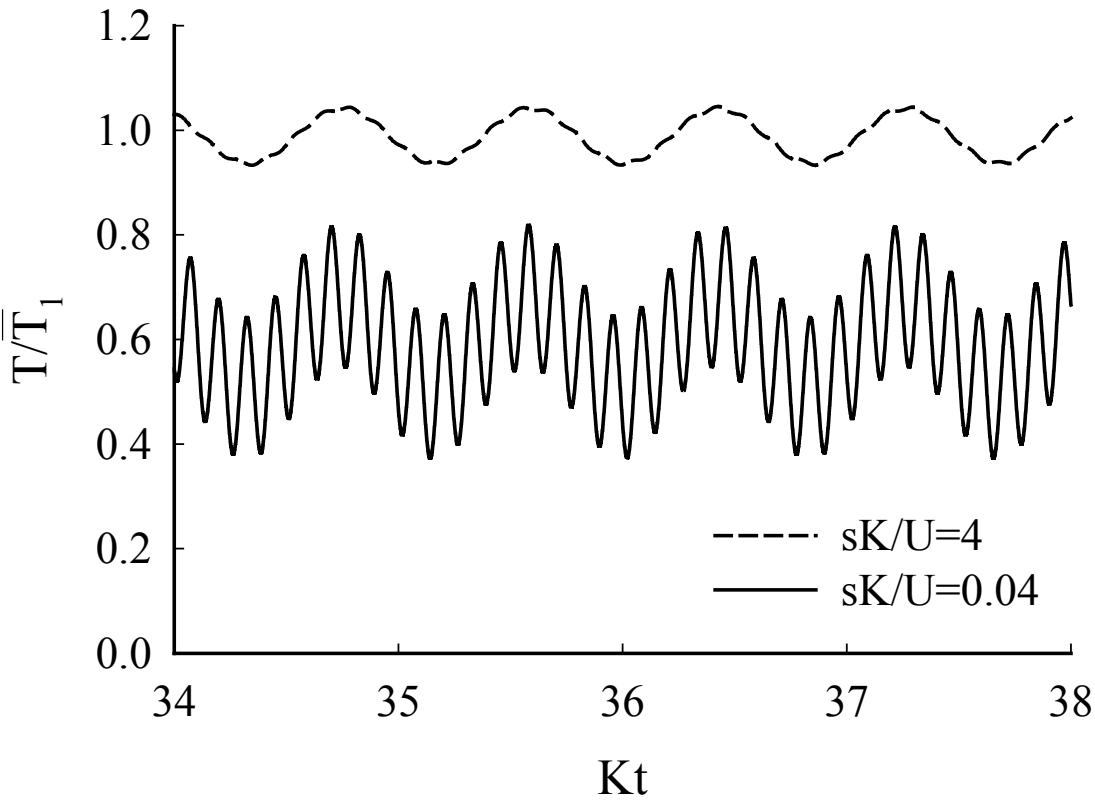


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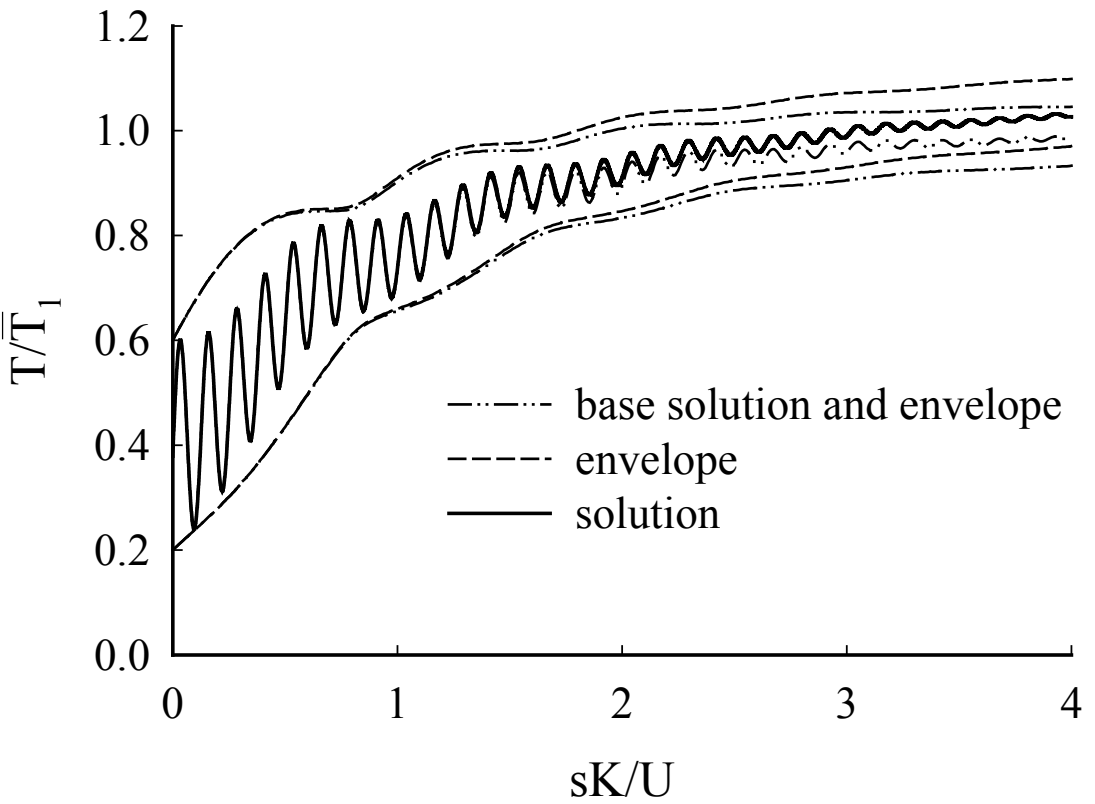




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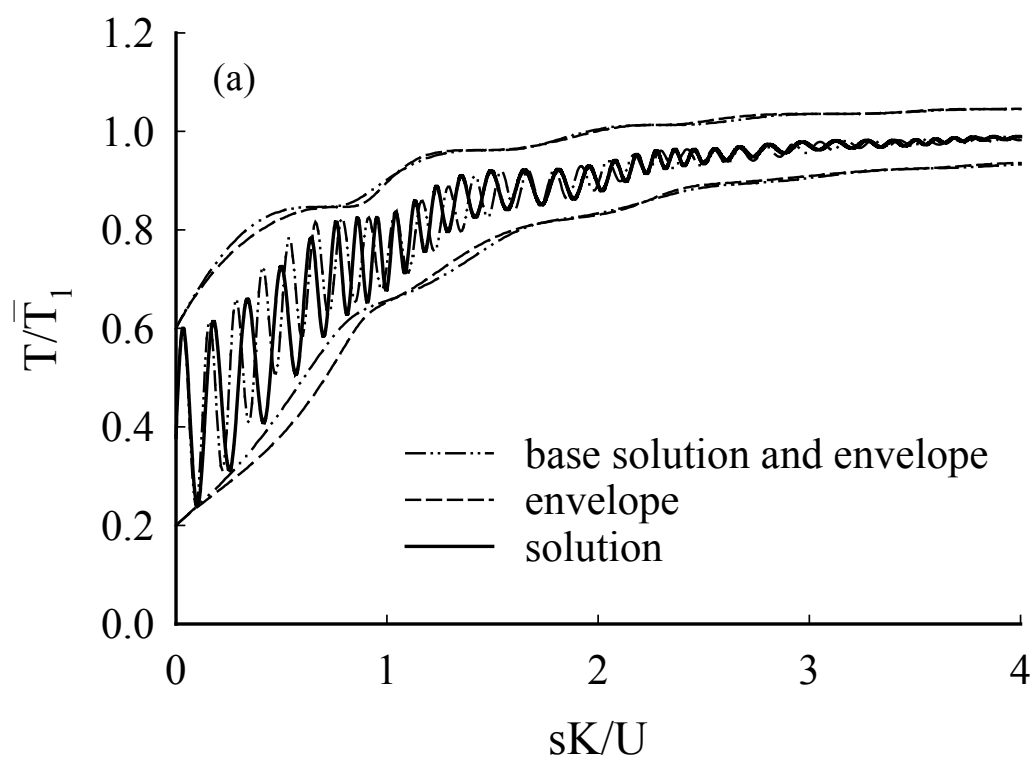


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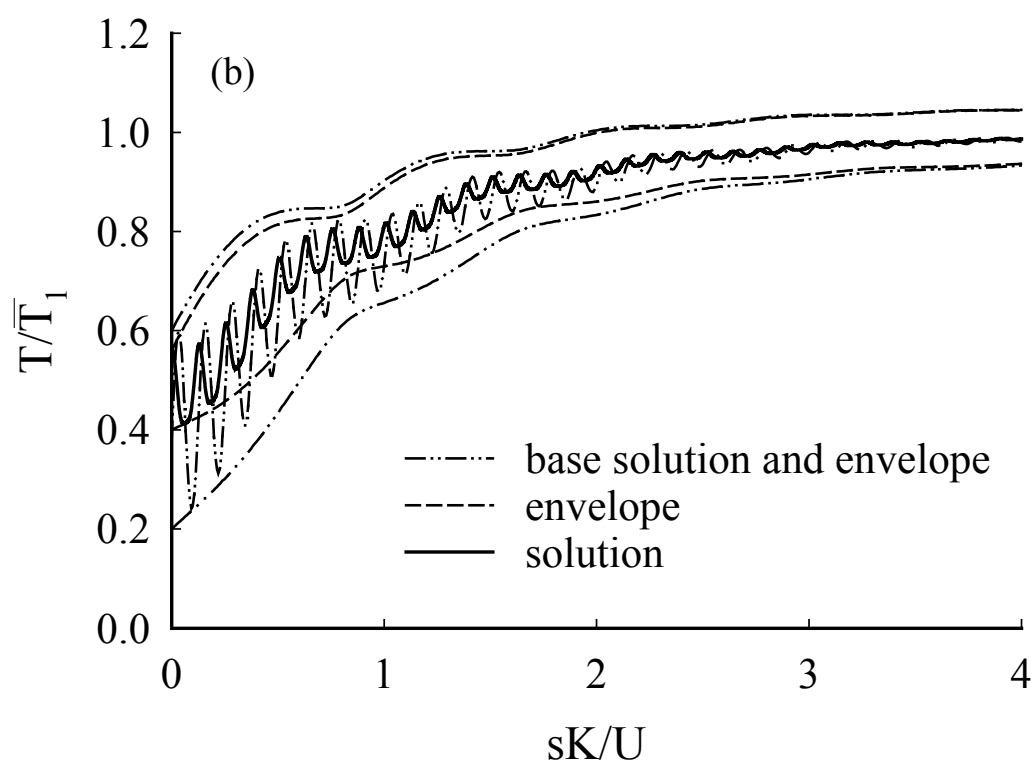


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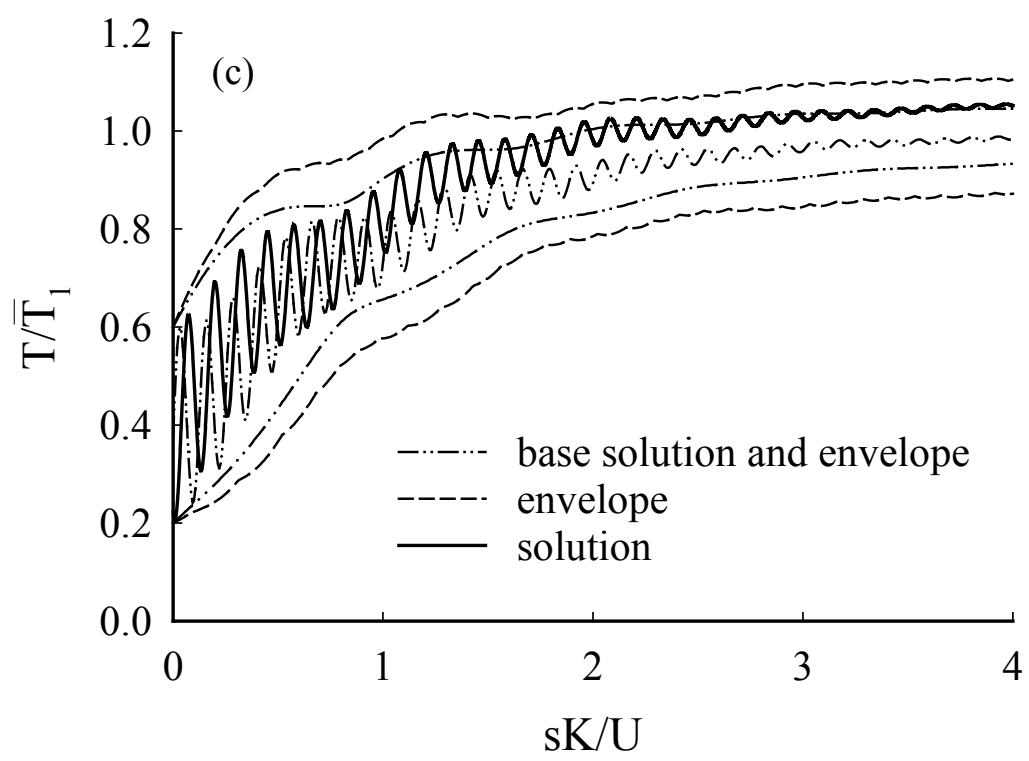


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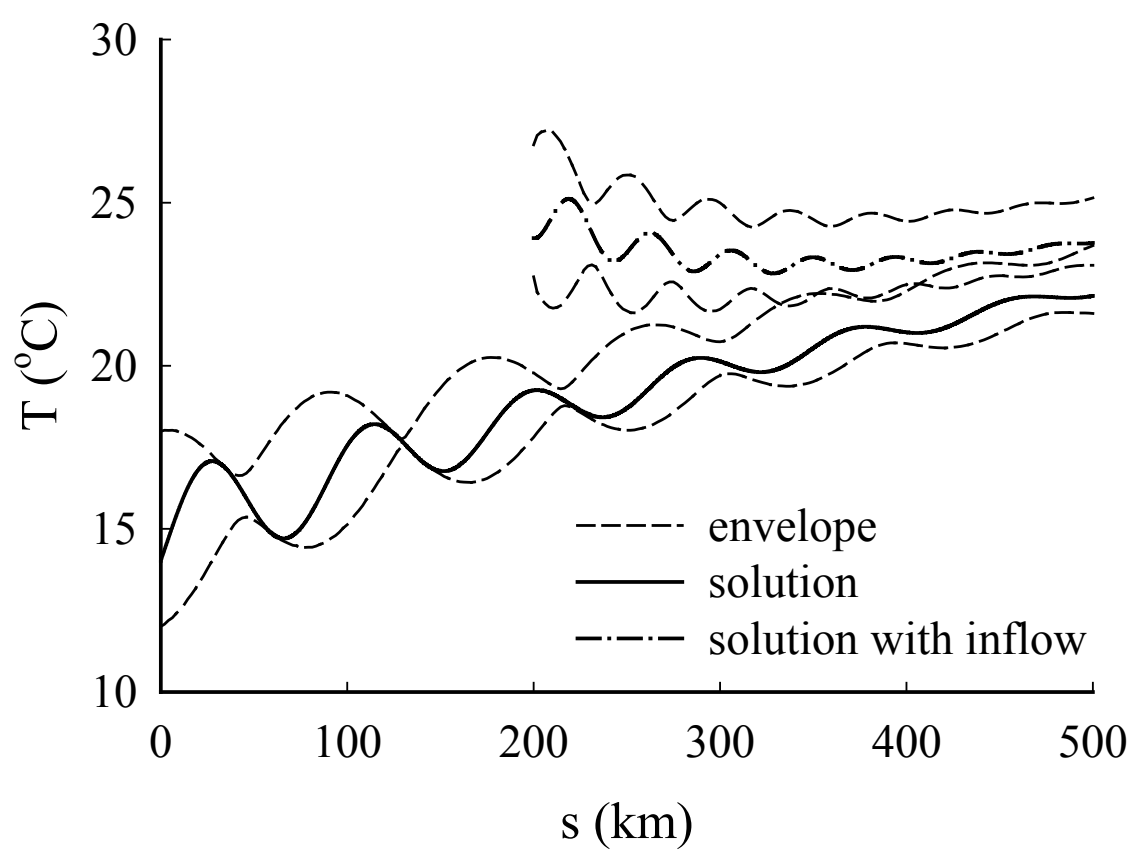


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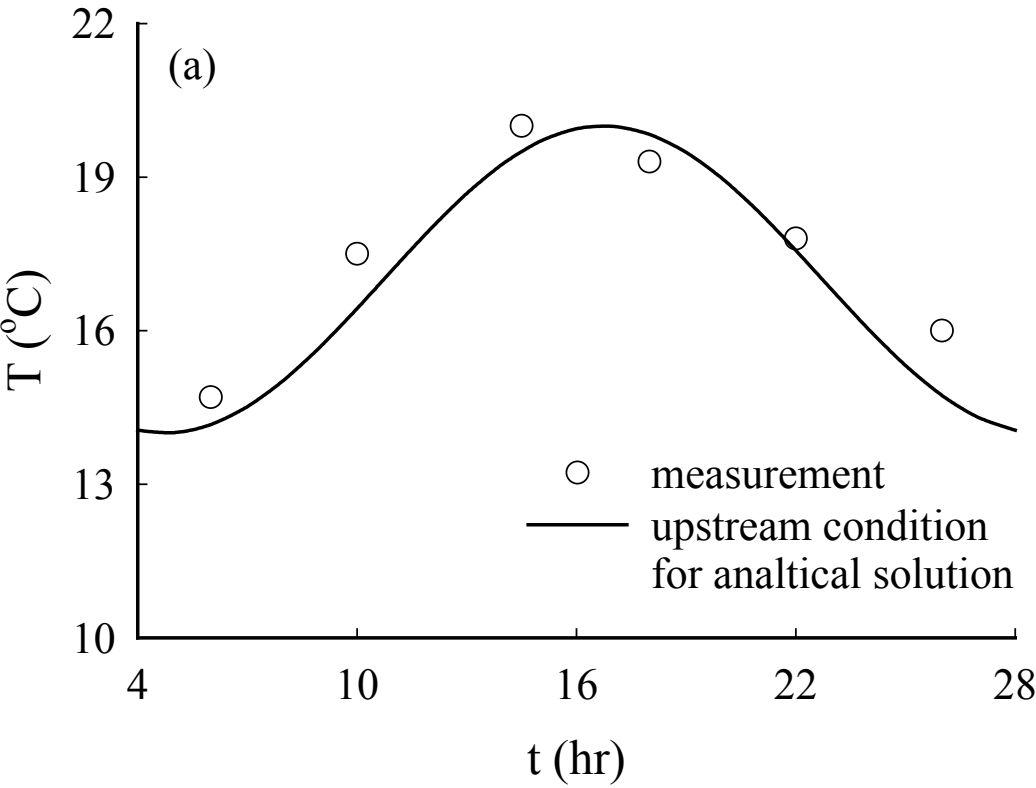


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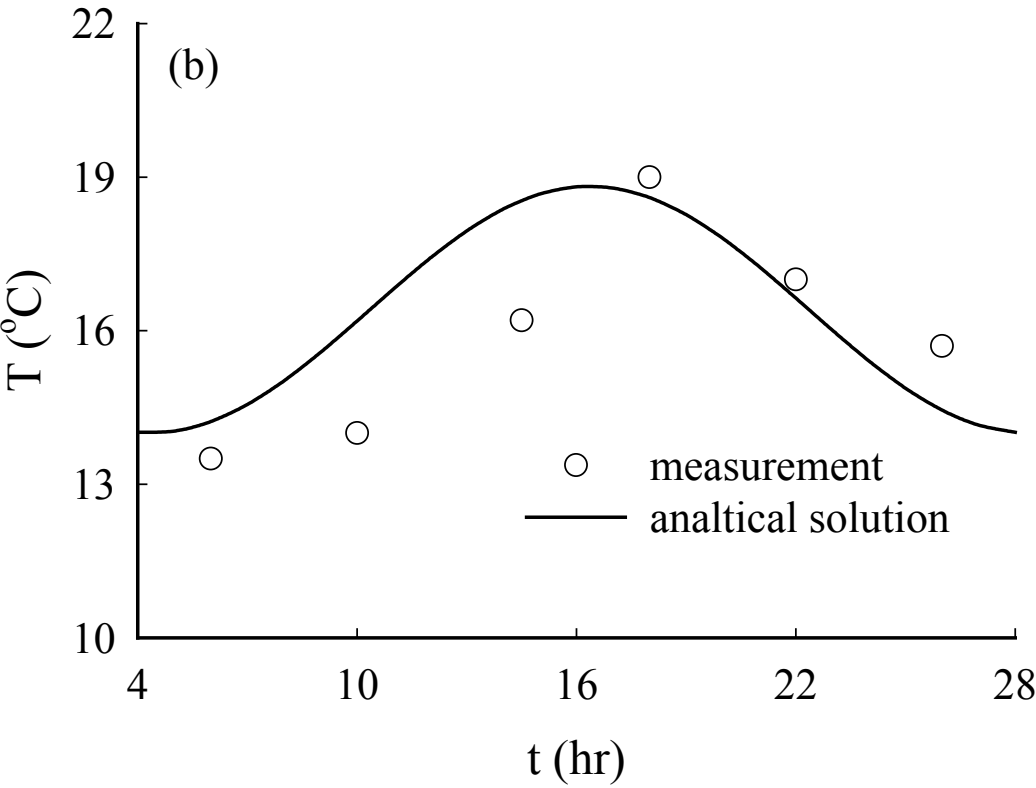


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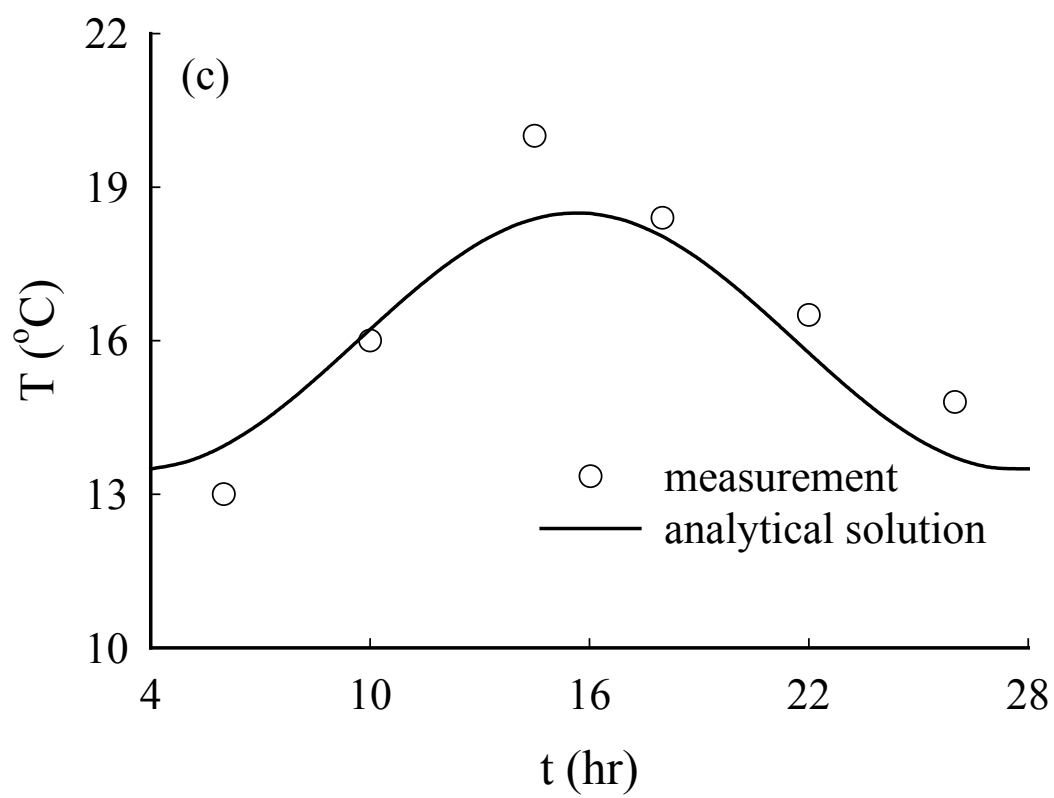


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